NUSC Technical Report 8759
6 August 1990

(2)

Two-Dimensional Convolutions, Correlations, and Fourier Transforms of Combinations of Wigner Distribution Functions and Complex Ambiguity Functions

Albert H. Nuttall
Surface ASW Directorate

Naval Underwater Systems Center
Newport, Rhode Island / New London, Connecticut

Preface

This research was conducted under NUSC Project No. A75215, Subproject No. R00N000, "Determination of Concentrated Energy Distribution Functions in the Time-Frequency Plane," Principal Investigator, Dr. Albert H. Nuttall (Code 304). This technical report was prepared with funds provided by the NUSC In-House Independent Research and Independent Exploratory Development Program, sponsored by the Office of the Chief of Naval Research. Also, this report was prepared under Project No. E65030, Subproject No. RV36I22, "Effect of Time Spread and Multipath on the Design of Waveforms for Matched Filters," Principal Investigator Roy L. Deavenport (Code 3112), sponsored by the Chief of Naval Technology, P. Quinn (Code 20T).

The technical reviewer for this report was Roy L. Deavenport (Code 3112).

Reviewed and Approved: 6 August 1990

Gerald M. Mayer (acting)

Larry Freeman
Associate Technical Director
Surface ASW Directorate

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
6 August 1990

3. REPORT TYPE AND DATES COVERED

4. TITLE AND SUBTITLE
TWO-DIMENSIONAL CONVOLUTIONS, CORRELATIONS, AND FOURIER TRANSFORMS OF COMBINATIONS OF WIGNER DISTRIBUTION FUNCTIONS AND COMPLEX AMBIGUITY FUNCTIONS

5. FUNDING NUMBERS
PE 61152N

6. AUTHOR(S)
Albert H. Nuttall

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Naval Underwater Systems Center
New London Laboratory
New London, CT 06320

8. PERFORMING ORGANIZATION REPORT NUMBER
TR 8759

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Office of the Chief of Naval Research
Arlington, VA 22217-5000

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT
A number of new two-dimensional Fourier transforms of combinations of cross Wigner distribution functions, W, of convolution form or correlation form are derived. In addition, similar relations are obtained for combinations of cross complex ambiguity functions, χ. Their great generality subsumes most of the already known available properties, such as: the volume constraint of magnitude-squared ambiguity functions; the positivity of the convolution of two Wigner distribution functions; and Moyal's theorem. An example is displayed below:

Keywords:

$$\iint d\nu' d\tau' \exp(+i2\pi\nu' t - i2\pi f\tau')\, \chi_{ab}(\nu+\tfrac{1}{2}\nu', \tau+\tfrac{1}{2}\tau')\, \chi^*_{cd}(\nu-\tfrac{1}{2}\nu', \tau-\tfrac{1}{2}\tau')$$

$$= \iint dt' df' \exp(-i2\pi\nu t' + i2\pi f'\tau)\, W_{ab}(t+\tfrac{1}{2}t', f+\tfrac{1}{2}f')\, W^*_{cd}(t-\tfrac{1}{2}t', f-\tfrac{1}{2}f')$$

14. SUBJECT TERMS
Two-dimensional, Convolution, Correlation, Temporal Correlation, Wigner Distribution, Ambiguity Function, Fourier Transform, Spectral Correlation. (KP)

15. NUMBER OF PAGES

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

13. ABSTRACT (Cont'd.)

$$= W_{ac}(t+\tfrac{1}{2}\tau, f+\tfrac{1}{2}\nu)\ W^{*}_{bd}(t-\tfrac{1}{2}\tau, f-\tfrac{1}{2}\nu)\ .$$

Extensions to contracted time and frequency arguments are made, as well as to mixed products involving a Wigner distribution function and a complex ambiguity function. Additional relationships connecting the temporal correlation function and the spectral correlation function complete a symmetric set of very general relationships.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF ABBREVIATIONS

TCF	temporal correlation function, (49)
SCF	spectral correlation function, (51)
CAF	complex ambiguity function, (53), (72)
WDF	Wigner distribution function, (55), (73)

LIST OF SYMBOLS

t	time, (1)
$g(t)$	arbitrary complex function of time, (1)
f	frequency, (1)
$G(f)$	Fourier transform of $g(t)$, $g(t) \leftrightarrow G(f)$, (1)
h,H	Fourier transform pair, (4)
ν	frequency shift or separation, (4), (27), (51)
α,β,μ,γ	real constants, (4)
x,X	Fourier transform pair, (10), (11)
ψ_{xx}	auto-correlation of x, (11)
C_{xy}	convolution of x and y, (16)

Ψ_{xy}	cross-correlation of x and y, (21)
R, W, χ, Φ	general two-dimensional functions, figure 1, (27)-(34)
τ	time delay or separation, (27), (49)
I	two-dimensional convolution and Fourier transform, (39)
J	two-dimensional correlation and Fourier transform, (43)
R_{ab}	cross temporal correlation function (TCF), (49)
Φ_{ab}	cross spectral correlation function (SCF), (51)
χ_{ab}	cross complex ambiguity function (CAF), (53), (72)
W_{ab}	cross Wigner distribution function (WDF), (55), (73)
$\underset{\sim}{W}_{ab}$	scaled and contracted WDF, (61)
Ψ_{ab}	cross-correlation of a(t) and b(t), (64)
Υ_{ab}	cross-spectrum of a(t) and b(t), (65)
$\underline{a}(t)$	mirror-image function of a(t), (69)
χ_{AB}	definition of CAF in frequency domain, (72)
W_{AB}	definition of WDF in frequency domain, (73)
$\zeta_n(t)$	n-th orthonormal Hermite function, (107)
W_{km}	cross WDF between ζ_k and ζ_m, (109)
α	contraction factor, (C-1)
$\underset{\sim}{a}(t)$	contracted waveform, (C-1)
K	more general two-dimensional transform, (C-3)
p	contraction parameter, (C-5)
$\mathbf{W}_{ab}$	generalized WDF, (C-5)

TWO-DIMENSIONAL CONVOLUTIONS, CORRELATIONS, AND FOURIER TRANSFORMS OF COMBINATIONS OF WIGNER DISTRIBUTION FUNCTIONS AND COMPLEX AMBIGUITY FUNCTIONS

INTRODUCTION

Over the years, a number of properties of integrals of products of complex ambiguity functions (CAFs) or products of Wigner distribution functions (WDFs) have been derived, such as: the volume constraint of magnitude-squared ambiguity functions [1; page 308], the positivity of the convolution of any two WDFs [2; (106)], and Moyal's theorem involving the volume under the square of a WDF [3]. Now, it appears that these are very special cases of a general class of two-dimensional Fourier transforms of combinations of CAFs and WDFs with delayed or time-reversed arguments.

We begin by deriving a general one-dimensional transform relation involving two arbitrary complex waveforms and their Fourier transforms. An application of this relation to energy density spectra yields three alternative expressions for the output correlation of a filtered time function. This general transform relation is also the basic tool for setting up the two-dimensional transforms that are the subject of succeeding sections. The extreme generality of the two-dimensional relations allows for a large number of special cases; some of these are pointed out, but undoubtedly there are additional ones not listed here.

When we begin our two-dimensional transform investigation, we do not immediately specialize to WDFs or CAFs. Rather, we first consider a set of four general functions, each of two variables, all of which are related to each other by Fourier transforms. We show that two-dimensional Fourier transforms of products of pairs of these general functions are all equal to a common value, although that value cannot be expressed in any simple closed form. These relations are derived for convolution type operations as well as for correlation operations.

When we make a specialization of these results to waveforms, relatively simple closed form results, in terms of products of WDFs and CAFs, are obtained for these two-dimensional transforms. And when the arguments of these relations are further specialized in value (such as zero), some of the currently known relations involving CAFs and WDFs result.

Extensions of these results to time contracted or expanded arguments are made in the appendices. Again, specializations to waveforms yield closed form results, in terms of products of WDFs and/or CAFs.

ONE-DIMENSIONAL TRANSFORM RELATIONS

Function g(t) is an arbitrary complex function of real argument t, which will be thought of as time. Its Fourier transform will be denoted by complex function G(f), where

$$G(f) = \int dt \exp(-i2\pi ft)\ g(t) \ . \tag{1}$$

Integrals without limits are along the real axis and over the range of nonzero integrand. Argument f is a real cyclic frequency, not a radian frequency. The inverse Fourier transform relation to (1) is

$$g(t) = \int df \exp(+i2\pi ft)\ G(f) \ . \tag{2}$$

The Fourier transform pair in (1) and (2) will be denoted by

$$g(t) \leftrightarrow G(f) \ . \tag{3}$$

Similarly, h(t) and H(f) will be a Fourier transform pair.

TRANSFORM OF PRODUCT OF WAVEFORMS

The variables $\nu,\alpha,\beta,\mu,\gamma$ are all real in the following. A generalization of Parseval's theorem is then possible, namely

$$\int dt' \exp(-i2\pi\nu t')\ g(\alpha t+\beta t')\ h^*(\mu t+\gamma t') = \exp\left(i2\pi\nu t\frac{\alpha\gamma+\beta\mu}{2\beta\gamma}\right) \times$$

$$\times \int d\nu' \exp\left(i2\pi\nu' t(\alpha\gamma-\beta\mu)\right)\ G\left(\gamma\left(\nu'+\frac{\nu}{2\beta\gamma}\right)\right)\ H^*\left(\beta\left(\nu'-\frac{\nu}{2\beta\gamma}\right)\right) \ , \tag{4}$$

where it is presumed that $\beta \neq 0$ and $\gamma \neq 0$. This result may be derived by substituting for g according to (2), interchanging integrals, and using (1) for Fourier transform pair $h(t) \leftrightarrow H(f)$. A more symmetric form for relation (4) is available, if desired:

$$\int dt' \exp(-i2\pi\nu t')\; g\left(\beta\left(t'+\frac{t}{2\beta\gamma}\right)\right)\; h^*\left(\gamma\left(t'-\frac{t}{2\beta\gamma}\right)\right) =$$

$$= \int d\nu' \exp(+i2\pi\nu' t)\; G\left(\gamma\left(\nu'+\frac{\nu}{2\beta\gamma}\right)\right)\; H^*\left(\beta\left(\nu'-\frac{\nu}{2\beta\gamma}\right)\right) . \qquad (5)$$

SPECIAL CASES

By specializing the parameter values in (4), several interesting and useful results can be obtained. For example, if we take $\gamma = \beta$, $\mu = -\alpha$, then we obtain a combined one-dimensional Fourier transform and correlation:

$$\int dt' \exp(-i2\pi\nu t')\; g(\beta t'+\alpha t)\; h^*(\beta t'-\alpha t) =$$

$$= \int d\nu' \exp\left(i2\pi\nu' t2\alpha\beta\right)\; G\left(\beta\nu'+\frac{\nu}{2\beta}\right)\; H^*\left(\beta\nu'-\frac{\nu}{2\beta}\right) . \qquad (6)$$

On the other hand, if we take $\gamma = -\beta$, $\mu = \alpha$ in (4), there follows a combined one-dimensional Fourier transform and convolution:

$$\int dt' \exp(-i2\pi\nu t')\; g(\alpha t+\beta t')\; h^*(\alpha t-\beta t') =$$

$$= \int d\nu' \exp\left(i2\pi\nu' t2\alpha\beta\right)\; G\left(\frac{\nu}{2\beta}+\beta\nu'\right)\; H^*\left(\frac{\nu}{2\beta}-\beta\nu'\right) . \qquad (7)$$

Further specialization to the specific numerical values $\gamma = \beta = 1$, $-\mu = \alpha = \frac{1}{2}$, in (6) yields

$$\int dt' \exp(-i2\pi\nu t')\, g(t'+\tfrac{1}{2}t)\, h^*(t'-\tfrac{1}{2}t) =$$

$$= \int d\nu' \exp(+i2\pi\nu' t)\, G(\nu'+\tfrac{1}{2}\nu)\, H^*(\nu'-\tfrac{1}{2}\nu) \ . \tag{8}$$

Alternatively, the choice $-\gamma = \beta = \frac{1}{2}$, $\mu = \alpha = 1$ in (7) yields

$$\int dt' \exp(-i2\pi\nu t')\, g(t+\tfrac{1}{2}t')\, h^*(t-\tfrac{1}{2}t') =$$

$$= \int d\nu' \exp(+i2\pi\nu' t)\, G(\nu+\tfrac{1}{2}\nu')\, H^*(\nu-\tfrac{1}{2}\nu') \ . \tag{9}$$

APPLICATION TO ENERGY DENSITY SPECTRA

Case 1. Suppose that we choose

$$G(\nu) = |X(\nu)|^2 \ , \quad H(\nu) = |Y(\nu)|^2 \ , \tag{10}$$

which are the energy density spectra of waveforms x(t) and y(t), respectively. Then $g(t) = \psi_{xx}(t)$ and $h(t) = \psi_{yy}(t)$, where $\psi_{xx}(t)$ is the auto-correlation function of complex waveform x(t):

$$\psi_{xx}(t) = \int du\, x(t + u)\, x^*(u) \ . \tag{11}$$

The use of (10) and (11) in (8) yields

$$I_1(t,\nu) \equiv \int d\nu' \exp(+i2\pi\nu' t) \, |X(\nu'+\tfrac{1}{2}\nu)|^2 \, |Y(\nu'-\tfrac{1}{2}\nu)|^2 =$$

$$= \int dt' \exp(-i2\pi\nu t') \, \psi_{xx}(t'+\tfrac{1}{2}t) \, \psi_{yy}^*(t'-\tfrac{1}{2}t) \ . \qquad (12)$$

The last term in (12) is identical to $\psi_{yy}(\tfrac{1}{2}t-t')$.

The special case of $\nu = 0$ in (12) reduces to

$$I_1(t,0) = \int d\nu' \exp(+i2\pi\nu' t) \, |X(\nu')|^2 \, |Y(\nu')|^2 =$$

$$= \int dt' \, \psi_{xx}(t'+\tfrac{1}{2}t) \, \psi_{yy}^*(t'-\tfrac{1}{2}t) \ . \qquad (13)$$

The additional restriction to $t = 0$ becomes

$$I_1(0,0) = \int d\nu' \, |X(\nu')|^2 \, |Y(\nu')|^2 =$$

$$= \int dt' \, \psi_{xx}(t') \, \psi_{yy}^*(t') \ . \qquad (14)$$

Case 2. Here, instead, make the identifications

$$G(\nu) = X(\nu) \, Y(\nu) = H(\nu) \ . \qquad (15)$$

Then

$$g(t) = C_{xy}(t) \equiv \int du \, x(u) \, y(t-u) = h(t) \ , \qquad (16)$$

which is the convolution of $x(t)$ and $y(t)$. Substitution of (15) and (16) in (8) gives

$$I_2(t,\nu) = \int d\nu' \exp(+i2\pi\nu' t)\, X(\nu'+\tfrac{1}{2}\nu)\, Y(\nu'+\tfrac{1}{2}\nu)\, X^*(\nu'-\tfrac{1}{2}\nu)\, Y^*(\nu'-\tfrac{1}{2}\nu)$$

$$= \int dt' \exp(-i2\pi\nu t')\, C_{xy}(t'+\tfrac{1}{2}t)\, C^*_{xy}(t'-\tfrac{1}{2}t) \; . \qquad (17)$$

Setting ν to zero yields

$$I_2(t,0) = \int d\nu' \exp(+i2\pi\nu' t)\, |X(\nu')|^2\, |Y(\nu')|^2 =$$

$$= \int dt'\, C_{xy}(t'+\tfrac{1}{2}t)\, C^*_{xy}(t'-\tfrac{1}{2}t) \; . \qquad (18)$$

Finally, also setting t equal to zero,

$$I_2(0,0) = \int d\nu'\, |X(\nu')|^2\, |Y(\nu')|^2 = \int dt'\, |C_{xy}(t')|^2 \; . \qquad (19)$$

Case 3. Now identify

$$G(\nu) = X(\nu)\, Y^*(\nu) = H(\nu) \; . \qquad (20)$$

Then

$$g(t) = \psi_{xy}(t) = \int du\, x(u+t)\, y^*(u) = h(t) \; , \qquad (21)$$

which is the cross-correlation of x(t) and y(t). The use of (20) and (21) in (8) leads to

$$I_3(t,\nu) = \int d\nu' \exp(+i2\pi\nu' t)\, X(\nu'+\tfrac{1}{2}\nu)\, Y^*(\nu'+\tfrac{1}{2}\nu)\, X^*(\nu'-\tfrac{1}{2}\nu)\, Y(\nu'-\tfrac{1}{2}\nu)$$

$$= \int dt' \exp(-i2\pi\nu t')\, \psi_{xy}(t'+\tfrac{1}{2}t)\, \psi^*_{xy}(t'-\tfrac{1}{2}t) \; . \qquad (22)$$

The result of setting ν to zero is

$$I_3(t,0) = \int d\nu' \exp(+i2\pi\nu' t) \, |X(\nu')|^2 \, |Y(\nu')|^2 =$$

$$= \int dt' \, \psi_{xy}(t'+\tfrac{1}{2}t) \, \psi^*_{xy}(t'-\tfrac{1}{2}t) \; . \qquad (23)$$

When t is also set equal to zero, (23) reduces to

$$I_3(0,0) = \int d\nu' \, |X(\nu')|^2 \, |Y(\nu')|^2 = \int dt' \, |\psi_{xy}(t')|^2 \; . \qquad (24)$$

It should be observed that the upper lines of (13), (18), and (23) are identical to each other; that is,

$$I_1(t,0) = I_2(t,0) = I_3(t,0) \; . \qquad (25)$$

Therefore, the lower lines of (13), (18), and (23) furnish three equal alternative expressions involving autocorrelations, convolutions, or cross-correlations, respectively.

There are many other possibilities for identifications of G and H in (8), besides (10), (15), and (20). For example, we could take

$$G(\nu) = |X(\nu)|^2 \, Y(\nu) \; , \quad H(\nu) = Y(\nu) \; . \qquad (26)$$

However, it may be shown that this choice leads identically to result (13) when ν is set to zero; so not all selections yield new relations. Additional convolution type relations may be obtained if (9) is used instead of (8).

GENERAL TWO-DIMENSIONAL TRANSFORM RELATIONS

In this section, we will consider a set of four general functions, each of two variables, which are related to each other by Fourier transforms. These four functions are indicated in figure 1, where a two-headed arrow denotes a Fourier transform relationship. These functions are, for the moment, <u>arbitrary</u> complex functions of two variables; they are <u>not</u> necessarily Wigner distribution functions or complex ambiguity functions.

$$\begin{tikzcd}
R(t,\tau) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \chi(\nu,\tau) \arrow[d, leftrightarrow] \\
W(t,f) \arrow[r, leftrightarrow] & \Phi(\nu,f)
\end{tikzcd}$$

Figure 1. General Two-Dimensional Functions

The paired transform variables, here and for the rest of the report, are $t \leftrightarrow \nu$ and $\tau \leftrightarrow f$. The detailed Fourier transform interrelationships between the four functions in figure 1 are

$$\chi(\nu,\tau) = \int dt \, \exp(-i2\pi\nu t) \, R(t,\tau) \, , \tag{27}$$

$$R(t,\tau) = \int d\nu \, \exp(+i2\pi\nu t) \, \chi(\nu,\tau) \, , \tag{28}$$

$$W(t,f) = \int d\tau \, \exp(-i2\pi f\tau) \, R(t,\tau) \, , \tag{29}$$

$$R(t,\tau) = \int df \, \exp(+i2\pi f\tau) \, W(t,f) \, , \tag{30}$$

$$\Phi(\nu,f) = \int dt \; \exp(-i2\pi\nu t) \; W(t,f) \; , \tag{31}$$

$$W(t,f) = \int d\nu \; \exp(+i2\pi\nu t) \; \Phi(\nu,f) \; , \tag{32}$$

$$\Phi(\nu,f) = \int d\tau \; \exp(-i2\pi f\tau) \; \chi(\nu,\tau) \; , \tag{33}$$

$$\chi(\nu,\tau) = \int df \; \exp(+i2\pi f\tau) \; \Phi(\nu,f) \; . \tag{34}$$

A double Fourier transform relationship exists between R and Φ, as well as between W and χ.

TWO-DIMENSIONAL CONVOLUTIONS

We repeat (9) here, but with a change of variables $t \to \tau$ and $\nu \to f$:

$$\int d\tau' \; \exp(-i2\pi f\tau') \; g(\tau+\tfrac{1}{2}\tau') \; h^*(\tau-\tfrac{1}{2}\tau') =$$

$$= \int df' \; \exp(+i2\pi f'\tau) \; G(f+\tfrac{1}{2}f') \; H^*(f-\tfrac{1}{2}f') \; . \tag{35}$$

Let χ_1 and χ_2 be two different functions of the type indicated in figure 1, and consider (35) with the assignments

$$g(\tau) = \chi_1(\nu_a,\tau) \; , \quad h(\tau) = \chi_2(\nu_b,\tau) \; . \tag{36}$$

The corresponding Fourier transform pairs for (36) are

$$G(f) = \Phi_1(\nu_a,f) \; , \quad H(f) = \Phi_2(\nu_b,f) \; , \tag{37}$$

upon use of (33). There follows, from (35),

$$\int d\tau' \, \exp(-i2\pi f\tau') \, \chi_1(\nu_a,\tau+\tfrac{1}{2}\tau') \, \chi_2^*(\nu_b,\tau-\tfrac{1}{2}\tau') =$$

$$= \int df' \, \exp(+i2\pi f'\tau) \, \Phi_1(\nu_a,f+\tfrac{1}{2}f') \, \Phi_2^*(\nu_b,f-\tfrac{1}{2}f') \; . \qquad (38)$$

See appendix A for the most general result of this form.

If we now let $\nu_a = \nu+\frac{1}{2}\nu'$ and $\nu_b = \nu-\frac{1}{2}\nu'$ in (38), then an additional Fourier transform on ν' yields the middle two lines in (39) below. More generally, in a similar fashion to that used above, we find that the combined two-dimensional convolution and Fourier transform can be expressed in four equivalent forms:

$$I(\nu,f,t,\tau) = \qquad (39)$$

$$= \iint dt'd\tau' \, \exp(-i2\pi\nu t'-i2\pi f\tau') \, R_1(t+\tfrac{1}{2}t',\tau+\tfrac{1}{2}\tau') \, R_2^*(t-\tfrac{1}{2}t',\tau-\tfrac{1}{2}\tau') =$$

$$= \iint d\nu'd\tau' \, \exp(+i2\pi\nu' t-i2\pi f\tau') \, \chi_1(\nu+\tfrac{1}{2}\nu',\tau+\tfrac{1}{2}\tau') \, \chi_2^*(\nu-\tfrac{1}{2}\nu',\tau-\tfrac{1}{2}\tau') =$$

$$= \iint d\nu'df' \, \exp(+i2\pi\nu' t+i2\pi f'\tau) \, \Phi_1(\nu+\tfrac{1}{2}\nu',f+\tfrac{1}{2}f') \, \Phi_2^*(\nu-\tfrac{1}{2}\nu',f-\tfrac{1}{2}f') =$$

$$= \iint dt'df' \, \exp(-i2\pi\nu t'+i2\pi f'\tau) \, W_1(t+\tfrac{1}{2}t',f+\tfrac{1}{2}f') \, W_2^*(t-\tfrac{1}{2}t',f-\tfrac{1}{2}f') \; .$$

Alternative forms of (39) are available; for example, the last line can be written in the more typical convolution form

$$\iint dt'df' \, \exp(-i2\pi\nu t'+i2\pi f'\tau) \, W_1(t',f') \, W_2^*(t-t',f-f') =$$

$$= \tfrac{1}{4} \exp(-i\pi\nu t+i\pi f\tau) \, I(\tfrac{1}{2}\nu,\tfrac{1}{2}f,\tfrac{1}{2}t,\tfrac{1}{2}\tau) \; . \qquad (40)$$

TWO-DIMENSIONAL CORRELATIONS

Here, we use (8) with identifications

$$g(t) = R_1(t,\tau_a) \ , \quad h(t) = R_2(t,\tau_b) \ ,$$
$$G(\nu) = X_1(\nu,\tau_a) \ , \quad H(\nu) = X_2(\nu,\tau_b) \ . \tag{41}$$

Then there follows immediately

$$\int dt' \exp(-i2\pi\nu t') \; R_1(t'+\tfrac{1}{2}t,\tau_a) \; R_2^*(t'-\tfrac{1}{2}t,\tau_b) =$$
$$= \int d\nu' \exp(+i2\pi\nu' t) \; X_1(\nu'+\tfrac{1}{2}\nu,\tau_a) \; X_2^*(\nu'-\tfrac{1}{2}\nu,\tau_b) \ . \tag{42}$$

Now let $\tau_a = \tau'+\frac{1}{2}\tau$ and $\tau_b = \tau'-\frac{1}{2}\tau$, and Fourier transform on τ'. The result is the first two relations, given below, of four equivalent forms of the combined two-dimensional correlation and Fourier transform

$$J(\nu,f,t,\tau) = \tag{43}$$

$$= \iint dt'd\tau' \exp(-i2\pi\nu t'-i2\pi f\tau') \; R_1(t'+\tfrac{1}{2}t,\tau'+\tfrac{1}{2}\tau) \; R_2^*(t'-\tfrac{1}{2}t,\tau'-\tfrac{1}{2}\tau) =$$

$$= \iint d\nu'd\tau' \exp(+i2\pi\nu' t-i2\pi f\tau') \; X_1(\nu'+\tfrac{1}{2}\nu,\tau'+\tfrac{1}{2}\tau) \; X_2^*(\nu'-\tfrac{1}{2}\nu,\tau'-\tfrac{1}{2}\tau) =$$

$$= \iint d\nu'df' \exp(+i2\pi\nu' t+i2\pi f'\tau) \; \Phi_1(\nu'+\tfrac{1}{2}\nu,f'+\tfrac{1}{2}f) \; \Phi_2^*(\nu'-\tfrac{1}{2}\nu,f'-\tfrac{1}{2}f) =$$

$$= \iint dt'df' \exp(-i2\pi\nu t'+i2\pi f'\tau) \; W_1(t'+\tfrac{1}{2}t,f'+\tfrac{1}{2}f) \; W_2^*(t'-\tfrac{1}{2}t,f'-\tfrac{1}{2}f) \ .$$

Alternative forms to (43) are possible; for example, the last line can be expressed in the more typical correlation form

$$\iint dt'df' \exp(-i2\pi\nu t'+i2\pi f'\tau)\, W_1(t',f')\, W_2^*(t'-t,f'-f) =$$

$$= \exp(-i\pi\nu t+i\pi f\tau)\, J(\nu,f,t,\tau)\ . \qquad (44)$$

MIXED RELATIONS

The results in (39) and (43) all involve two $W(t,f)$ functions, or two $\chi(\nu,\tau)$ functions, etc. However, it is possible to obtain relations which involve, for example, one $W(t,f)$ function and one $\chi(\nu,\tau)$ function. As an illustrative example, consider (9) with $g(t) = W_1(t,f_a)$ and $h(t) = \chi_2(f_b,t)$. Then, from figure 1, $G(\nu) = \Phi_1(\nu,f_a)$ and $H(\nu) = \Phi_2(f_b,\nu)$, giving

$$\int dt' \exp(-i2\pi\nu t')\, W_1(t+\tfrac{1}{2}t',f_a)\, \chi_2^*(f_b,t-\tfrac{1}{2}t') =$$

$$= \int d\nu' \exp(+i2\pi\nu' t)\, \Phi_1(\nu+\tfrac{1}{2}\nu',f_a)\, \Phi_2^*(f_b,\nu-\tfrac{1}{2}\nu')\ . \qquad (45)$$

If we now let $f_a = f+\tfrac{1}{2}f'$ and $f_b = f-\tfrac{1}{2}f'$, and perform a Fourier transform on f', there follows immediately

$$\iint dt'df' \exp(-i2\pi\nu t'+i2\pi f'\tau)\, W_1(t+\tfrac{1}{2}t',f+\tfrac{1}{2}f')\, \chi_2^*(f-\tfrac{1}{2}f',t-\tfrac{1}{2}t') =$$

$$(46)$$

$$= \iint d\nu'df' \exp(+i2\pi\nu' t+i2\pi f'\tau)\, \Phi_1(\nu+\tfrac{1}{2}\nu',f+\tfrac{1}{2}f')\, \Phi_2^*(f-\tfrac{1}{2}f',\nu-\tfrac{1}{2}\nu')\ .$$

Thus, a combined two-dimensional convolution and Fourier transform of a $W(t,f)$ function and a $\chi(\nu,\tau)$ function can be expressed in terms of two $\Phi(\nu,f)$ functions. (Strictly, some of the arguments are reversed, as seen in (46).)

If, instead, we use (8) with g(t) and h(t) assigned as above, then we obtain

$$\int dt' \exp(-i2\pi\nu t')\, W_1(t'+\tfrac{1}{2}t,f_a)\, \chi_2^*(f_b,t'-\tfrac{1}{2}t) =$$

$$= \int d\nu' \exp(+i2\pi\nu' t)\, \Phi_1(\nu'+\tfrac{1}{2}\nu,f_a)\, \Phi_2^*(f_b,\nu'-\tfrac{1}{2}\nu) \ . \qquad (47)$$

Letting $f_a = f'+\frac{1}{2}f$, $f_b = f'-\frac{1}{2}f$, and performing an additional Fourier transform on f', there follows

$$\iint dt'df' \exp(-i2\pi\nu t'+i2\pi f'\tau)\, W_1(t'+\tfrac{1}{2}t,f'+\tfrac{1}{2}f)\, \chi_2^*(f'-\tfrac{1}{2}f,t'-\tfrac{1}{2}t) =$$

$$(48)$$

$$= \iint d\nu'df' \exp(+i2\pi\nu' t+i2\pi f'\tau)\, \Phi_1(\nu'+\tfrac{1}{2}\nu,f'+\tfrac{1}{2}f)\, \Phi_2^*(f'-\tfrac{1}{2}f,\nu'-\tfrac{1}{2}\nu) \ .$$

Here, a combined two-dimensional correlation and Fourier transform of a $W(t,f)$ function and a $\chi(\nu,\tau)$ function can be expressed in terms of two $\Phi(\nu,f)$ functions. (Again, some arguments are reversed or replaced. However, the first argument in a χ function is always a frequency variable, while the second argument is always a time variable; similar restrictions hold for the remaining functions R, W, Φ in figure 1.)

SPECIALIZATION TO WAVEFORMS

In the previous section, the functions R, W, X, Φ were arbitrary, except that they were related by Fourier transforms according to figure 1. Here, we will specialize their forms, thereby enabling more explicit relations for their two-dimensional convolutions and correlations.

For arbitrary complex waveforms a(t), b(t), c(t), d(t), let

$$R_1(t,\tau) = a(t+\tfrac{1}{2}\tau)\ b^*(t-\tfrac{1}{2}\tau) \equiv R_{ab}(t,\tau) \ , \tag{49}$$

$$R_2(t,\tau) = c(t+\tfrac{1}{2}\tau)\ d^*(t-\tfrac{1}{2}\tau) \equiv R_{cd}(t,\tau) \ . \tag{50}$$

These are known as (cross) temporal correlation functions (TCFs). Thus, $R_{ab}(t,\tau)$ is the "instantaneous" cross-correlation between waveforms a and b, corresponding to center time t and separation (or delay) time τ. Then, from (31) and (29), or [4; (35)], there follows

$$\Phi_1(\nu,f) = \Phi_{ab}(\nu,f) = \iint dt\ d\tau\ \exp(-i2\pi\nu t - i2\pi f\tau)\ R_{ab}(t,\tau) =$$

$$= A(f+\tfrac{1}{2}\nu)\ B^*(f-\tfrac{1}{2}\nu) \ , \tag{51}$$

$$\Phi_2(\nu,f) = \Phi_{cd}(\nu,f) = C(f+\tfrac{1}{2}\nu)\ D^*(f-\tfrac{1}{2}\nu) \ . \tag{52}$$

These functions are known as (cross) spectral correlation functions (SCFs). (In [4], the notation $A(\nu,f)$ was used for this function; however, A(f) will be used here for the Fourier transform of waveform a(t).) The SCF corresponds to center frequency f and separation (or shift) frequency ν.

The Fourier transform relationships in figure 1 and equations (27) - (34) still hold true, but now are specialized to the waveform cases above. Specifically, figure 2 illustrates the four two-dimensional functions for waveforms a(t) and b(t), where now $W_1 = W_{ab}$ is a cross Wigner distribution function (WDF) and $X_1 = X_{ab}$ is a cross complex ambiguity function (CAF).

$$
\begin{array}{cccc}
\text{TCF} & R_{ab}(t,\tau) \longleftrightarrow & X_{ab}(\nu,\tau) & \text{CAF} \\
& \updownarrow & \updownarrow & \\
\text{WDF} & W_{ab}(t,f) \longleftrightarrow & \Phi_{ab}(\nu,f) & \text{SCF}
\end{array}
$$

Figure 2. Two-Dimensional Functions for Waveforms

The detailed Fourier transform interrelationships are now

$$X_{ab}(\nu,\tau) = \int dt \; \exp(-i2\pi\nu t) \; R_{ab}(t,\tau) \; , \tag{53}$$

$$R_{ab}(t,\tau) = \int d\nu \; \exp(+i2\pi\nu t) \; X_{ab}(\nu,\tau) \; , \tag{54}$$

$$W_{ab}(t,f) = \int d\tau \; \exp(-i2\pi f\tau) \; R_{ab}(t,\tau) \; , \tag{55}$$

$$R_{ab}(t,\tau) = \int df \; \exp(+i2\pi f\tau) \; W_{ab}(t,f) \; , \tag{56}$$

$$\Phi_{ab}(\nu,f) = \int dt \; \exp(-i2\pi\nu t) \; W_{ab}(t,f) \; , \tag{57}$$

$$W_{ab}(t,f) = \int d\nu \; \exp(+i2\pi\nu t) \; \Phi_{ab}(\nu,f) \; , \tag{58}$$

$$\Phi_{ab}(\nu,f) = \int d\tau \; \exp(-i2\pi f\tau) \; X_{ab}(\nu,\tau) \; , \tag{59}$$

$$X_{ab}(\nu,\tau) = \int df \; \exp(+i2\pi f\tau) \; \Phi_{ab}(\nu,f) \; . \tag{60}$$

The function $W_{aa}(t,f)$, for example, is an auto WDF, since it involves only one waveform, $a(t)$. We will frequently drop the terminology auto and cross, when possible without confusion, and let the notation indicate the particular case.

It will be found advantageous for future purposes to define a scaled and contracted WDF according to

$$\underset{\sim}{W}_{ab}(t,f) = \tfrac{1}{2} W_{ab}(\tfrac{1}{2}t,\tfrac{1}{2}f) \; . \tag{61}$$

GENERAL CROSS PROPERTIES

Due to the restriction of form taken on by the TCF in (49) and the SCF in (51), the four functions in figure 2 obey some symmetry rules; they are

$$\begin{aligned} R_{ab}(t,-\tau) &= R^{*}_{ba}(t,\tau) \; , \\ \Phi_{ab}(-\nu,f) &= \Phi^{*}_{ba}(\nu,f) \; , \\ X_{ab}(-\nu,-\tau) &= X^{*}_{ba}(\nu,\tau) \; , \\ W_{ab}(t,f) &= W^{*}_{ba}(t,f) \; . \end{aligned} \tag{62}$$

AUTO PROPERTIES

When waveform $b(t) = a(t)$, some specializations follow:

$$R_{aa}(t,-\tau) = R^*_{aa}(t,\tau) \; ,$$

$$\Phi_{aa}(-\nu,f) = \Phi^*_{aa}(\nu,f) \; ,$$

$$X_{aa}(-\nu,-\tau) = X^*_{aa}(\nu,\tau) \; ,$$

$$W_{aa}(t,f) = \text{real for all } t,\ f,\ a(t), \qquad (63)$$

with the only significant specialization being the realness of WDF $W_{aa}(t,f)$. Waveform $a(t)$ can still be complex.

SOME SPECIAL CASES

The ordinary cross-correlation of two waveforms $a(t)$ and $b(t)$ is a special case of a CAF:

$$\psi_{ab}(\tau) = \int dt\; a(t)\; b^*(t-\tau) = X_{ab}(0,\tau) \; . \qquad (64)$$

The ordinary cross-spectrum is then a special case of an SCF:

$$\Psi_{ab}(f) = \int d\tau\; \exp(-i2\pi f\tau)\; \psi_{ab}(\tau) = \Phi_{ab}(0,f) = A(f)\; B^*(f) \; . \qquad (65)$$

The autospectrum is then simply

$$\Psi_{aa}(f) = \Phi_{aa}(0,f) = |A(f)|^2 \; , \qquad (66)$$

which is always nonnegative.

The ordinary convolution of two waveforms $a(t)$ and $b(t)$ is a special case of a WDF:

$$\int d\tau\; a(\tau)\; b^*(t-\tau) = \tfrac{1}{2} W_{ab}(\tfrac{1}{2}t,0) = \underset{\sim}{W}_{ab}(t,0) \; . \qquad (67)$$

REAL WAVEFORM a(t)

In addition, if waveform a(t) is real, the following (auto) properties hold true:

$$R_{aa}(t,-\tau) = R_{aa}(t,\tau) \quad \text{and } R_{aa} \text{ is real} ,$$

$$\Phi_{aa}(\nu,-f) = \Phi_{aa}(\nu,f) ,$$

$$X_{aa}(\nu,-\tau) = X_{aa}(\nu,\tau) ,$$

$$W_{aa}(t,-f) = W_{aa}(t,f) . \tag{68}$$

The situation for a real waveform a(t) is summarized in figure 3 below.

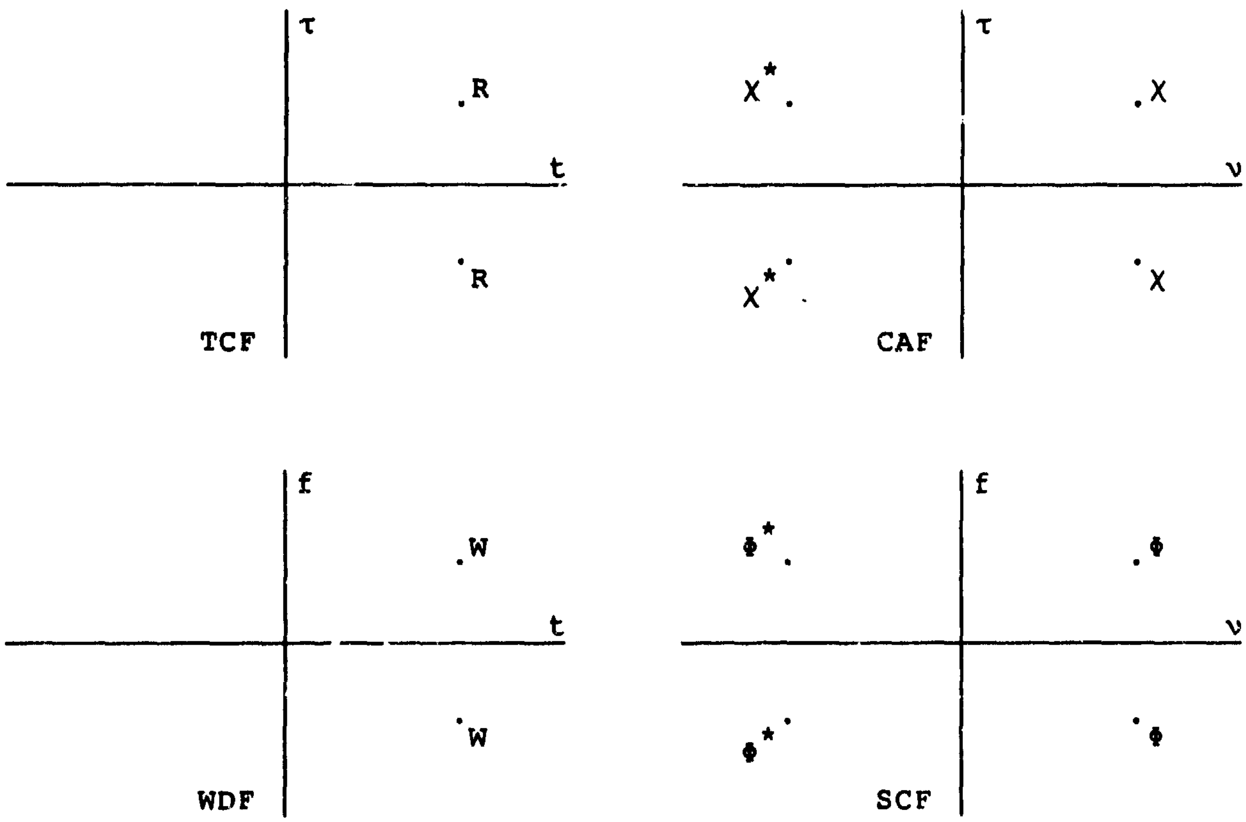

Figure 3. Symmetry Properties for Real Waveform a(t)

MIRROR-IMAGE RELATIONS

For general complex waveforms a(t) and b(t), define mirror-image functions

$$\underline{a}(t) = a(-t) , \qquad \underline{b}(t) = b(-t) . \tag{69}$$

Then it follows directly that the voltage density spectrum of mirror-image $\underline{a}(t)$ is

$$\underline{A}(f) = \int dt\, \exp(-i2\pi ft)\, \underline{a}(t) = A(-f) , \tag{70}$$

which is the mirror-image of A(f). Also, there follows

$$\begin{aligned} R_{ab}(-t,-\tau) &= R_{\underline{ab}}(t,\tau) , \\ \Phi_{ab}(-\nu,-f) &= \Phi_{\underline{ab}}(\nu,f) , \\ X_{ab}(-\nu,-\tau) &= X_{\underline{ab}}(\nu,\tau) , \\ W_{ab}(-t,-f) &= W_{\underline{ab}}(t,f) . \end{aligned} \tag{71}$$

Thus, the mirror-image property for A(f) carries over into all the two-dimensional domains, such as the WDF and CAF, as well. There is no significant simplification for b(t) = a(t), except for the realness of $W_{aa}(t,f)$, as before.

Use of mirror-image definition (69) allows for an interesting connection between WDFs and CAFs. First, substituting (49) into (53) and (55), we have cross CAF

$$\chi_{ab}(\nu,\tau) = \int dt\ \exp(-i2\pi\nu t)\ a(t+\tfrac{1}{2}\tau)\ b^*(t-\tfrac{1}{2}\tau) =$$

$$= \int df\ \exp(+i2\pi f\tau)\ A(f+\tfrac{1}{2}\nu)\ B^*(f-\tfrac{1}{2}\nu) \equiv \chi_{AB}(\nu,\tau) \tag{72}$$

and cross WDF

$$W_{ab}(t,f) = \int d\tau\ \exp(-i2\pi f\tau)\ a(t+\tfrac{1}{2}\tau)\ b^*(t-\tfrac{1}{2}\tau) =$$

$$= \int d\nu\ \exp(+i2\pi\nu t)\ A(f+\tfrac{1}{2}\nu)\ B^*(f-\tfrac{1}{2}\nu) \equiv W_{AB}(t,f)\ . \tag{73}$$

Reference to (69) now immediately reveals that

$$W_{ab}(t,f) = 2\chi_{a\underline{b}}(2f,2t) \tag{74}$$

or

$$\chi_{ab}(\nu,\tau) = \tfrac{1}{2}W_{a\underline{b}}(\tfrac{1}{2}\tau,\tfrac{1}{2}\nu) = \underset{\sim}{W}_{a\underline{b}}(\tau,\nu)\ . \tag{75}$$

Here, we also used (61). That is, the WDF of two waveforms a and b is proportional to the CAF of waveforms a and $\underline{b}$, the mirror-image of b.

Finally, since

$$B^*(f) \leftrightarrow b^*(-t) = \underline{b}^*(t)\ , \tag{76}$$

then, using (72),

$$\chi_{AB^*}(\nu,\tau) = \int df\ \exp(i2\pi f\tau)\ A(f+\tfrac{1}{2}\nu)\ B(f-\tfrac{1}{2}\nu) =$$

$$= \chi_{a\underline{b}^*}(\nu,\tau) = \tfrac{1}{2}W_{ab^*}(\tfrac{1}{2}\tau,\tfrac{1}{2}\nu) = \underset{\sim}{W}_{ab^*}(\tau,\nu)\ . \tag{77}$$

TWO-DIMENSIONAL TRANSFORM RELATIONS FOR WAVEFORMS

In an earlier section, general two-dimensional transform relations were derived between sets of four functions related by Fourier transforms; see figure 1 and (39) and (43). Here, we will utilize the particular forms considered in the previous section for waveforms (see figure 2) and will derive closed forms for I and J in (39) and (43), respectively.

TWO-DIMENSIONAL CONVOLUTIONS

If we substitute (49) and (50) in the top relation in (39), there follows

$$I(\nu,f,t,\tau) = \iint dt'd\tau' \exp(-i2\pi\nu t'-i2\pi f\tau')\, a(t+\tfrac{1}{2}t'+\tfrac{1}{2}\tau+\tfrac{1}{4}\tau') \times$$

$$\times\, b^*(t+\tfrac{1}{2}t'-\tfrac{1}{2}\tau-\tfrac{1}{4}\tau') \times c^*(t-\tfrac{1}{2}t'+\tfrac{1}{2}\tau-\tfrac{1}{4}\tau')\, d(t-\tfrac{1}{2}t'-\tfrac{1}{2}\tau+\tfrac{1}{4}\tau') \; . \qquad (78)$$

Now let

$$u = \tfrac{1}{2}t'+\tfrac{1}{4}\tau', \; v = \tfrac{1}{2}t'-\tfrac{1}{4}\tau'; \qquad u+v = t', \; 2(u-v) = \tau'. \qquad (79)$$

Since the Jacobian of this transformation is 4, (78) becomes

$$I(\nu,f,t,\tau) = 4\iint du\, dv\, \exp\Big(-i2\pi\nu(u+v)-i2\pi f2(u-v)\Big) \times$$

$$\times\, a(t+\tfrac{1}{2}\tau+u)\, b^*(t-\tfrac{1}{2}\tau+v)\, c^*(t+\tfrac{1}{2}\tau-u)\, d(t-\tfrac{1}{2}\tau-v) =$$

$$= \int du' \exp(-i2\pi u'(f+\tfrac{1}{2}\nu))\, a(t+\tfrac{1}{2}\tau+\tfrac{1}{2}u')\, c^*(t+\tfrac{1}{2}\tau-\tfrac{1}{2}u') \times$$

$$\times \int dv' \exp(+i2\pi v'(f-\tfrac{1}{2}\nu))\, b^*(t-\tfrac{1}{2}\tau+\tfrac{1}{2}v')\, d(t-\tfrac{1}{2}\tau-\tfrac{1}{2}v') =$$

$$= W_{ac}(t+\tfrac{1}{2}\tau, f+\tfrac{1}{2}\nu)\ W^{*}_{bd}(t-\tfrac{1}{2}\tau, f-\tfrac{1}{2}\nu)\ . \tag{80}$$

That is, all the following quantities are equal:

$$I(\nu,f,t,\tau) =$$

$$= \iint dt'd\tau'\ \exp(-i2\pi\nu t'-i2\pi f\tau')\ R_{ab}(t+\tfrac{1}{2}t',\tau+\tfrac{1}{2}\tau')\ R^{*}_{cd}(t-\tfrac{1}{2}t',\tau-\tfrac{1}{2}\tau') =$$

$$= \iint d\nu'd\tau'\ \exp(+i2\pi\nu' t-i2\pi f\tau')\ \chi_{ab}(\nu+\tfrac{1}{2}\nu',\tau+\tfrac{1}{2}\tau')\ \chi^{*}_{cd}(\nu-\tfrac{1}{2}\nu',\tau-\tfrac{1}{2}\tau') =$$

$$= \iint d\nu'df'\ \exp(+i2\pi\nu' t+i2\pi f'\tau)\ \Phi_{ab}(\nu+\tfrac{1}{2}\nu',f+\tfrac{1}{2}f')\ \Phi^{*}_{cd}(\nu-\tfrac{1}{2}\nu',f-\tfrac{1}{2}f') =$$

$$= \iint dt'df'\ \exp(-i2\pi\nu t'+i2\pi f'\tau)\ W_{ab}(t+\tfrac{1}{2}t',f+\tfrac{1}{2}f')\ W^{*}_{cd}(t-\tfrac{1}{2}t',f-\tfrac{1}{2}f') =$$

$$= W_{ac}(t+\tfrac{1}{2}\tau, f+\tfrac{1}{2}\nu)\ W^{*}_{bd}(t-\tfrac{1}{2}\tau, f-\tfrac{1}{2}\nu)\ . \tag{81}$$

All four double-integrals in (81) can be expressed as a product of the same two one-dimensional integrals, which are cross WDFs. This reduction is only possible when the two-dimensional functions, like W_{ab} and χ_{ab}, are WDFs and CAFs, respectively. The transformations in (81) are combined two-dimensional Fourier transforms and convolutions of TCFs, CAFs, SCFs, or WDFs.

By use of (74), an alternative expression for the end result in (81) is

$$I(\nu,f,t,\tau) = 4\ \chi_{a\underline{c}}(2f+\nu,2t+\tau)\ \chi^{*}_{b\underline{d}}(2f-\nu,2t-\tau)\ , \tag{82}$$

in terms of mirror-image functions; see (69). Also, a more typical convolution form for (81), for example, is (using (61))

$$\iint du\, dv\, \exp(-i2\pi\nu u+i2\pi\nu\tau)\, W_{ab}(u,v)\, W_{cd}^{*}(t-u,f-v) =$$

$$= \exp(-i\pi\nu t+i\pi f\tau)\, \underset{\sim}{W}_{ac}(t+\tfrac{1}{2}\tau,f+\tfrac{1}{2}\nu)\, \underset{\sim}{W}_{bd}^{*}(t-\tfrac{1}{2}\tau,f-\tfrac{1}{2}\nu)\ . \qquad (83)$$

TWO-DIMENSIONAL CORRELATIONS

In an identical fashion to that used above, result (43) becomes

$$J(\nu,f,t,\tau) =$$

$$= \iint dt'd\tau'\, \exp(-i2\pi\nu t'-i2\pi f\tau')\, R_{ab}(t'+\tfrac{1}{2}t,\tau'+\tfrac{1}{2}\tau)\, R_{cd}^{*}(t'-\tfrac{1}{2}t,\tau'-\tfrac{1}{2}\tau) =$$

$$= \iint d\nu'd\tau'\, \exp(+i2\pi\nu' t-i2\pi f\tau')\, X_{ab}(\nu'+\tfrac{1}{2}\nu,\tau'+\tfrac{1}{2}\tau)\, X_{cd}^{*}(\nu'-\tfrac{1}{2}\nu,\tau'-\tfrac{1}{2}\tau) =$$

$$= \iint d\nu'df'\, \exp(+i2\pi\nu' t+i2\pi f'\tau)\, \Phi_{ab}(\nu'+\tfrac{1}{2}\nu,f'+\tfrac{1}{2}f)\, \Phi_{cd}^{*}(\nu'-\tfrac{1}{2}\nu,f'-\tfrac{1}{2}f) =$$

$$= \iint dt'df'\, \exp(-i2\pi\nu t'+i2\pi f'\tau)\, W_{ab}(t'+\tfrac{1}{2}t,f'+\tfrac{1}{2}f)\, W_{cd}^{*}(t'-\tfrac{1}{2}t,f'-\tfrac{1}{2}f) =$$

$$= X_{ac}(f+\tfrac{1}{2}\nu,t+\tfrac{1}{2}\tau)\, X_{bd}^{*}(f-\tfrac{1}{2}\nu,t-\tfrac{1}{2}\tau)\ . \qquad (84)$$

All these double integrals in (84) are equal to a product of two cross CAFs. Again, this only holds for the special forms of the two-dimensional functions, like W_{ab} and X_{ab}, which are WDFs and CAFs, respectively. The transformations in (84) are combined two-dimensional Fourier transforms and correlations of TCFs, CAFs, SCFs, or WDFs.

By use of (75), an alternative expression for the end result in (84) is

$$J(\nu,f,t,\tau) = \underset{\sim}{W}_{a\underline{c}}(t+\tfrac{1}{2}\tau,f+\tfrac{1}{2}\nu)\ \underset{\sim}{W}^{*}_{b\underline{d}}(t-\tfrac{1}{2}\tau,f-\tfrac{1}{2}\nu)\ , \tag{85}$$

in terms of mirror-image functions. Also, a more typical correlation form for (84) is, for example,

$$\iint du\ dv\ \exp(-i2\pi\nu u+i2\pi v\tau)\ W_{ab}(u,v)\ W^{*}_{cd}(u-t,v-f) =$$

$$= \exp(-i\pi\nu t+i\pi f\tau)\ X_{ac}(f+\tfrac{1}{2}\nu,t+\tfrac{1}{2}\tau)\ X^{*}_{bd}(f-\tfrac{1}{2}\nu,t-\tfrac{1}{2}\tau)\ . \tag{86}$$

A MIXED RELATION

As an example in this category, if we take (46) with

$$W_1(t,f) = W_{ab}(t,f)\ ,\qquad X_2(\nu,\tau) = X_{cd}(2\nu,2\tau)\ , \tag{87}$$

then

$$\Phi_1(\nu,f) = \Phi_{ab}(\nu,f) = A(f+\tfrac{1}{2}\nu)\ B^{*}(f-\tfrac{1}{2}\nu)\ ,$$

$$\Phi_2(\nu,f) = \tfrac{1}{2}\ \Phi_{cd}(2\nu,\tfrac{1}{2}f) = \tfrac{1}{2}\ C(\tfrac{1}{2}f+\nu)\ D^{*}(\tfrac{1}{2}f-\nu)\ . \tag{88}$$

Substitution of these results in (46) yields

$$\iint dt'df'\ \exp(-i2\pi\nu t'+i2\pi f'\tau)\ W_{ab}(t+\tfrac{1}{2}t',f+\tfrac{1}{2}f')\ X^{*}_{cd}(2f-f',2t-t') =$$

$$= W_{ac}(t+\tfrac{1}{2}\tau,f+\tfrac{1}{2}\nu)\ X^{*}_{bd}(2f-\nu,2t-\tau)\ . \tag{89}$$

This mixed relation is a two-dimensional Fourier transform and convolution, involving a WDF and a CAF, expressible in closed form as a product of another WDF and CAF.

SPECIAL CASES

The two-dimensional transform results in (81) and (84) in the previous section involve four arguments, namely ν, f, t, τ, and four functions, $a(t)$, $b(t)$, $c(t)$, $d(t)$. Their extreme generality allows for numerous special cases upon selection of the arguments and/or the functions. We consider some of these possibilities, but are aware that this list could be considerably augmented.

<u>Case 1.</u> As an example of the generality of these results, consider in (84) the particular selection

$$\nu = f = t = \tau = 0, \quad c(t) = a(t), \quad d(t) = b(t). \tag{90}$$

There follows immediately the "volume constraint"

$$\iint d\nu' d\tau' \left|\chi_{ab}(\nu',\tau')\right|^2 = \iint dt' df' \left|W_{ab}(t',f')\right|^2 =$$

$$= \chi_{aa}(0,0)\ \chi_{bb}(0,0) = \int dt\ |a(t)|^2 \int dt\ |b(t)|^2 . \tag{91}$$

<u>Case 2.</u> In (84), take $\nu = \tau = 0$, $b(t) = a(t)$, $d(t) = c(t)$. Then there follows, upon use of (85),

$$\iint d\nu' d\tau' \exp(+i2\pi\nu' t - i2\pi f\tau')\ \chi_{aa}(\nu',\tau')\ \chi^{*}_{cc}(\nu',\tau') =$$

$$= \iint dt' df'\ W_{aa}(t'+\tfrac{1}{2}t, f'+\tfrac{1}{2}f)\ W_{cc}(t'-\tfrac{1}{2}t, f'-\tfrac{1}{2}f) =$$

$$= \left|\chi_{ac}(f,t)\right|^2 = \left|\underset{\sim}{W}_{a\underline{c}}(t,f)\right|^2 , \tag{92}$$

which is nonnegative real for all f, t, a(t), c(t). Thus, the two-dimensional correlation of two auto WDFs is nonnegative.

An alternative form of (92) is

$$\iint du\, dv\; W_{aa}(u,v)\; W_{cc}(u-t,v-f) = \left|X_{ac}(f,t)\right|^2 . \qquad (93)$$

Further specialization to t = f = 0 yields

$$\iint du\, dv\; W_{aa}(u,v)\; W_{cc}(u,v) = \left|X_{ac}(0,0)\right|^2 = \left|\int dt\; a(t)\; c^*(t)\right|^2, \quad (94)$$

which yields Moyal's result [3] for c(t) = a(t), namely

$$\iint dt\, df\; W_{aa}^2(t,f) = \left[\int dt\; |a(t)|^2\right]^2 . \qquad (95)$$

<u>Case 3.</u> In (81), take $\nu = \tau = 0$, b(t) = a(t), d(t) = c(t). We then get the "smoothing result"

$$\iint dt'df'\; W_{aa}(t+\tfrac{1}{2}t',f+\tfrac{1}{2}f')\; W_{cc}(t-\tfrac{1}{2}t',f-\tfrac{1}{2}f') =$$

$$= \left|W_{ac}(t,f)\right|^2 = \left|\int d\tau'\; \exp(-i2\pi f\tau')\; a(t+\tfrac{1}{2}\tau')\; c^*(t-\tfrac{1}{2}\tau')\right|^2 \geq 0 \quad (96)$$

for all t, f, a(t), c(t). An alternative form is

$$\iint du\, dv\; W_{aa}(u,v)\; W_{cc}(t-u,f-v) = \left|\tfrac{1}{2}W_{ac}(\tfrac{1}{2}t,\tfrac{1}{2}f)\right|^2 = \left|\underset{\sim}{W}_{ac}(t,f)\right|^2 =$$

$$= \left|\int d\tau'\; \exp(-i2\pi f\tau')\; a(\tau')\; c^*(t-\tau')\right|^2 . \qquad (97)$$

That is, the two-dimensional convolution of two auto WDFs is never negative (just as for the correlation in (92)).

Case 4. Using (62), the same basic end result is obtained from (81) for the following double integral involving CAFs:

$$\iint d\nu' d\tau' \exp(+i2\pi\nu' t - i2\pi f\tau')\, \chi_{aa}(\tfrac{1}{2}\nu', \tfrac{1}{2}\tau')\, \chi_{cc}(\tfrac{1}{2}\nu', \tfrac{1}{2}\tau') =$$

$$= \left| W_{ac}(t,f) \right|^2 . \tag{98}$$

This right-hand side is nonnegative real for all t, f, a(t), c(t). An alternative form is, upon use of (61),

$$\iint d\nu\, d\tau \exp(+i2\pi\nu t - i2\pi f\tau)\, \chi_{aa}(\nu,\tau)\, \chi_{cc}(\nu,\tau) = \left| \underset{\sim}{W}_{ac}(t,f) \right|^2 . \tag{99}$$

Case 5. Consider (81) with c(t) = a(t), d(t) = b(t). Then the right-hand side of (81) is always real. For example, we have

$$\iint d\nu' d\tau' \exp(+i2\pi\nu' t - i2\pi f\tau')\, \chi_{ab}(\nu+\tfrac{1}{2}\nu', \tau+\tfrac{1}{2}\tau')\, \chi^*_{ab}(\nu-\tfrac{1}{2}\nu', \tau-\tfrac{1}{2}\tau') =$$

$$= \iint dt' df' \exp(-i2\pi\nu t' + i2\pi f'\tau)\, W_{ab}(t+\tfrac{1}{2}t', f+\tfrac{1}{2}f')\, W^*_{ab}(t-\tfrac{1}{2}t', f-\tfrac{1}{2}f') =$$

$$= W_{aa}(t+\tfrac{1}{2}\tau, f+\tfrac{1}{2}\nu)\, W_{bb}(t-\tfrac{1}{2}\tau, f-\tfrac{1}{2}\nu) . \tag{100}$$

This is real for all t, τ, f, ν, a(t), b(t), although it could go negative.

Case 6. From (81), with ν = τ = 0, there follows

$$\iint dt' df'\, W_{ab}(t+\tfrac{1}{2}t', f+\tfrac{1}{2}f')\, W^*_{cd}(t-\tfrac{1}{2}t', f-\tfrac{1}{2}f') =$$

$$= W_{ac}(t,f)\, W^*_{bd}(t,f) , \tag{101}$$

or, with the help of (61) and (75), alternative form

$$\iint du\, dv\, W_{ab}(u,v)\, W^*_{cd}(t-u,f-v) =$$

$$= \underset{\sim}{W}_{ac}(t,f)\, \underset{\sim}{W}^*_{bd}(t,f) = \chi_{a\underline{c}}(f,t)\, \chi^*_{b\underline{d}}(f,t) \; . \tag{102}$$

Furthermore, if we set $c(t) = a(t)$, $d(t) = b(t)$, we obtain

$$\iint du\, dv\, W_{ab}(u,v)\, W^*_{ab}(t-u,f-v) =$$

$$= \underset{\sim}{W}_{aa}(t,f)\, \underset{\sim}{W}_{bb}(t,f) = \chi_{a\underline{a}}(f,t)\, \chi^*_{b\underline{b}}(f,t) \; . \tag{103}$$

Thus, the two-dimensional convolution of a complex cross WDF with itself is always real, but could go negative.

<u>Case 7.</u> From (83) and (84), with $\nu = \tau = 0$, there follows

$$\iint dt'df'\, W_{ab}(t'+\tfrac{1}{2}t,f'+\tfrac{1}{2}f)\, W^*_{cd}(t'-\tfrac{1}{2}t,f'-\tfrac{1}{2}t) =$$

$$= \iint du\, dv\, W_{ab}(u,v)\, W^*_{cd}(u-t,v-f) =$$

$$= \iint d\nu' d\tau'\, \exp(+i2\pi\nu' t - i2\pi f\tau')\, \chi_{ab}(\nu',\tau')\, \chi^*_{cd}(\nu',\tau') =$$

$$= \chi_{ac}(f,t)\, \chi^*_{bd}(f,t) = \underset{\sim}{W}_{a\underline{c}}(t,f)\, \underset{\sim}{W}^*_{b\underline{d}}(t,f) \; . \tag{104}$$

The two-dimensional correlation of two cross WDFs is a product of two cross CAFs.

<u>Case 8.</u> If we now set $c(t) = a(t)$ and $d(t) = b(t)$ in (104), we obtain

$$\iint du\, dv\, W_{ab}(u,v)\, W_{ab}^{*}(u-t,v-f) =$$

$$= \iint d\nu' d\tau' \exp(+i2\pi\nu' t - i2\pi f\tau') \left|\chi_{ab}(\nu',\tau')\right|^2 =$$

$$= \chi_{aa}(f,t)\, \chi_{bb}^{*}(f,t) = \underset{\sim}{W}_{aa}(t,f)\, \underset{\sim}{W}_{bb}^{*}(t,f) \quad . \qquad (105)$$

The two-dimensional correlation of a cross WDF with itself is a product of two auto CAFs.

<u>Case 9.</u> From (84), with $t = f = 0$, $c(t) = a(t)$, $d(t) = b(t)$, and with the help of (63), we find

$$\iint dt' df' \exp(-i2\pi\nu t' + i2\pi f'\tau) \left|W_{ab}(t',f')\right|^2 =$$

$$= \chi_{aa}(\tfrac{1}{2}\nu,\tfrac{1}{2}\tau)\, \chi_{bb}(\tfrac{1}{2}\nu,\tfrac{1}{2}\tau) \quad . \qquad (106)$$

This is a generalization of (91).

APPLICATION TO HERMITE FUNCTIONS

This material is heavily based on [5; appendix A, (A-36) and the sequel]. Let $\zeta_n(t)$ be the n-th orthonormal Hermite function with linear frequency-modulation, as given in [5; (A-36)]. Also let waveforms

$$a(t) = \zeta_k(\mu t),\ b(t) = \zeta_l(\gamma t),\ c(t) = \zeta_m(\mu t),\ d(t) = \zeta_n(\gamma t). \quad (107)$$

The particular cross WDFs

$$W_{ab}(t,f) = \int d\tau\ \exp(-i2\pi f\tau)\ \zeta_k(\mu t+\tfrac{1}{2}\mu\tau)\ \zeta_l^*(\gamma t-\tfrac{1}{2}\gamma\tau)\ ,$$

$$W_{cd}(t,f) = \int d\tau\ \exp(-i2\pi f\tau)\ \zeta_m(\mu t+\tfrac{1}{2}\mu\tau)\ \zeta_n^*(\gamma t-\tfrac{1}{2}\gamma\tau)\ , \quad (108)$$

cannot be expressed in closed form. However, the cross WDFs

$$W_{ac}(t,f) = \int d\tau\ \exp(-i2\pi f\tau)\ \zeta_k(\mu t+\tfrac{1}{2}\mu\tau)\ \zeta_m^*(\mu t-\tfrac{1}{2}\mu\tau) =$$

$$= \frac{1}{\mu}\ W_{km}(\mu t, f/\mu) \quad (109)$$

and

$$W_{bd}(t,f) = \frac{1}{\gamma}\ W_{ln}(\gamma t, f/\gamma) \quad (110)$$

can be simply expressed, in the notation of [5; (A-40) and (A-41)]. Thus, the very complicated two-dimensional convolution and Fourier transform in (81), of W_{ab} and W_{cd}, can be written in a closed form involving the product of two generalized Laguerre functions. Numerous specializations are possible.

SUMMARY

Some very general two-dimensional Fourier transforms of convolution and correlation form have been derived for various combinations of WDFs and CAFs. In particular, closed forms for the convolution form are given in (81), while results for the correlation form are given in (84). Numerous special cases may be obtained from these results, of which a brief list has been presented in (90) - (106).

Some extensions to more general arguments have been derived in appendices A and B. In particular, appendix A treats the case where a product of CAFs is of interest, while the case of a product of WDFs is considered in appendix B. The possibility of a combined convolution and correlation has also been considered in appendix A.

For signals reflected off moving targets, it is necessary to define a generalized WDF, allowing for contracted arguments. This possibility has been considered in appendix C, where a two-dimensional Fourier transform and convolution has been evaluated in terms of the generalized WDF.

The results of this report should enable rapid evaluation of integrals of products of WDFs and/or CAFs with a wide variety of arguments and including exponential terms with linear arguments. They also significantly extend a number of special cases already known in the literature.

APPENDIX A - PRODUCTS OF CAFs

In this appendix, we will further generalize the results in (81) and (84), for products of two CAFs, to allow for more general arguments. However, we begin by considering general two-dimensional functions as in figure 1. In particular, let

$$g(\tau) = \chi_1(\nu_a,\tau) \ , \quad h(\tau) = \chi_2(\nu_b,\tau) \ , \tag{A-1}$$

in (4). Then

$$G(f) = \Phi_1(\nu_a,f) \ , \quad H(f) = \Phi_2(\nu_b,f) \ , \tag{A-2}$$

giving

$$\int d\tau' \ \exp(-i2\pi f\tau') \ \chi_1(\nu_a,\beta\tau'+\alpha\tau) \ \chi_2^*(\nu_b,\gamma\tau'+\mu\tau) = \exp\left(i2\pi f\tau\frac{\alpha\gamma+\beta\mu}{2\beta\gamma}\right) \times$$

$$\times \int df' \ \exp\left(i2\pi f'\tau(\alpha\gamma-\beta\mu)\right) \ \Phi_1\left(\nu_a,\gamma\left(f'+\frac{f}{2\beta\gamma}\right)\right) \ \Phi_2^*\left(\nu_b,\beta\left(f'-\frac{f}{2\beta\gamma}\right)\right) =$$

$$= \frac{1}{|\alpha\gamma-\beta\mu|} \exp\left(i2\pi f\tau\frac{\alpha\gamma+\beta\mu}{2\beta\gamma}\right) \int df' \ \exp(i2\pi f'\tau) \ \times$$

$$\times \ \Phi_1\left(\nu_a, \frac{\gamma f'}{\alpha\gamma-\beta\mu} + \frac{f}{2\beta}\right) \ \Phi_2^*\left(\nu_b, \frac{\beta f'}{\alpha\gamma-\beta\mu} - \frac{f}{2\gamma}\right) \ . \tag{A-3}$$

Now, let $\nu_a=\boldsymbol{\beta}\nu'+\boldsymbol{\alpha}\nu$, $\nu_b=\boldsymbol{\gamma}\nu'+\boldsymbol{\mu}\nu$, where the boldface constants are unrelated to their counterparts; that is, $\boldsymbol{\beta}$ need not equal β, with the same true of $\boldsymbol{\alpha},\boldsymbol{\mu},\boldsymbol{\gamma}$. Then Fourier transform (A-3) on ν' to obtain

$$\iint d\nu' d\tau' \exp(+i2\pi\nu' t - i2\pi f\tau')\ \chi_1(\beta\nu'+\alpha\nu, \beta\tau'+\alpha\tau)\ \chi_2^*(\gamma\nu'+\mu\nu, \gamma\tau'+\mu\tau) =$$

$$= \frac{1}{|\alpha\gamma-\beta\mu|} \exp\left(i2\pi f\tau \frac{\alpha\gamma+\beta\mu}{2\beta\gamma}\right) \iint d\nu' df' \exp(+i2\pi\nu' t + i2\pi f'\tau) \times$$

$$\times\ \Phi_1\left(\beta\nu'+\alpha\nu,\ \frac{\gamma f'}{\alpha\gamma-\beta\mu} + \frac{f}{2\beta}\right) \Phi_2^*\left(\gamma\nu'+\mu\nu,\ \frac{\beta f'}{\alpha\gamma-\beta\mu} - \frac{f}{2\gamma}\right) . \qquad \text{(A-4)}$$

In general, we cannot proceed any further on this double integral of a product of general two-dimensional functions χ_1 and χ_2.

Now let R_1 and R_2 be TCFs; that is,

$$R_1(t,\tau) = a(t+\tfrac{1}{2}\tau)\ b^*(t-\tfrac{1}{2}\tau) = R_{ab}(t,\tau) ,$$

$$R_2(t,\tau) = c(t+\tfrac{1}{2}\tau)\ d^*(t-\tfrac{1}{2}\tau) = R_{cd}(t,\tau) . \qquad \text{(A-5)}$$

Then Φ_1 and Φ_2 become SCFs:

$$\Phi_1(\nu,f) = \Phi_{ab}(\nu,f) = A(f+\tfrac{1}{2}\nu)\ B^*(f-\tfrac{1}{2}\nu) ,$$

$$\Phi_2(\nu,f) = \Phi_{cd}(\nu,f) = C(f+\tfrac{1}{2}\nu)\ D^*(f-\tfrac{1}{2}\nu) . \qquad \text{(A-6)}$$

As a first case, let $\gamma=\beta$ and $\gamma=\beta$. Then (A-4) becomes

$$\iint d\nu' d\tau' \exp(i2\pi\nu' t - i2\pi f\tau')\ \chi_{ab}(\beta\nu'+\alpha\nu, \beta\tau'+\alpha\tau)\ \chi_{cd}^*(\beta\nu'+\mu\nu, \beta\tau'+\mu\tau) =$$

$$= \frac{1}{|\beta(\alpha-\mu)|} \exp\left(i2\pi f\tau \frac{\alpha+\mu}{2\beta}\right) \iint d\nu' df' \exp(+i2\pi\nu' t + i2\pi f'\tau) \times$$

$$\times\ A\left(\frac{f'}{\alpha-\mu} + \tfrac{1}{2}\frac{f}{\beta} + \tfrac{1}{2}\beta\nu' + \tfrac{1}{2}\alpha\nu\right) B^*\left(\frac{f'}{\alpha-\mu} + \tfrac{1}{2}\frac{f}{\beta} - \tfrac{1}{2}\beta\nu' - \tfrac{1}{2}\alpha\nu\right) \times$$

$$\times\ C^*\left(\frac{f'}{\alpha-\mu} - \tfrac{1}{2}\frac{f}{\beta} + \tfrac{1}{2}\beta\nu' + \tfrac{1}{2}\mu\nu\right) D\left(\frac{f'}{\alpha-\mu} - \tfrac{1}{2}\frac{f}{\beta} - \tfrac{1}{2}\beta\nu' - \tfrac{1}{2}\mu\nu\right) =$$

$$= |\beta\beta|^{-1} \exp\left(+i2\pi f\tau\frac{\alpha+\mu}{2\beta} - i2\pi\nu t\frac{\alpha+\mu}{2\beta}\right) \times$$

$$\times\, X_{ac}\left(\frac{f}{\beta} + \tfrac{1}{2}\nu(\alpha-\mu), \frac{t}{\beta} + \tfrac{1}{2}\tau(\alpha-\mu)\right) X^{*}_{bd}\left(\frac{f}{\beta} - \tfrac{1}{2}\nu(\alpha-\mu), \frac{t}{\beta} - \tfrac{1}{2}\tau(\alpha-\mu)\right). \quad \text{(A-7)}$$

Thus, this very general two-dimensional correlation and Fourier transform of cross CAFs can be expressed as a product of two different cross CAFs. For $\beta=\beta=1$, $\alpha=\alpha=\frac{1}{2}$, $\mu=\mu=-\frac{1}{2}$, this result reduces to (84).

As a second case, let $\gamma=-\beta$ and $\gamma=-\beta$. Then (A-4) becomes

$$\iint d\nu' d\tau' \exp(i2\pi\nu' t - i2\pi f\tau')\; X_{ab}(\alpha\nu+\beta\nu', \alpha\tau+\beta\tau')\; X^{*}_{cd}(\mu\nu-\beta\nu', \mu\tau-\beta\tau') =$$

$$= \frac{1}{|\beta(\alpha+\mu)|} \exp\left(i2\pi f\tau\frac{\alpha-\mu}{2\beta}\right) \iint d\nu' df' \exp(+i2\pi\nu' t + i2\pi f'\tau) \times$$

$$\times\, A\left(\frac{f'}{\alpha+\mu} + \tfrac{1}{2}\frac{f}{\beta} + \tfrac{1}{2}\beta\nu' + \tfrac{1}{2}\alpha\nu\right) B^{*}\left(\frac{f'}{\alpha+\mu} + \tfrac{1}{2}\frac{f}{\beta} - \tfrac{1}{2}\beta\nu' - \tfrac{1}{2}\alpha\nu\right) \times$$

$$\times\, C^{*}\left(\frac{-f'}{\alpha+\mu} + \tfrac{1}{2}\frac{f}{\beta} - \tfrac{1}{2}\beta\nu' + \tfrac{1}{2}\mu\nu\right) D\left(\frac{-f'}{\alpha+\mu} + \tfrac{1}{2}\frac{f}{\beta} + \tfrac{1}{2}\beta\nu' - \tfrac{1}{2}\mu\nu\right) =$$

$$= |\beta\beta|^{-1} \exp\left(+i2\pi f\tau\frac{\alpha-\mu}{2\beta} - i2\pi\nu t\frac{\alpha-\mu}{2\beta}\right) \times$$

$$\times\, \underset{\sim}{W}_{ac}\left(\frac{t}{\beta} + \tfrac{1}{2}\tau(\alpha+\mu), \frac{f}{\beta} + \tfrac{1}{2}\nu(\alpha+\mu)\right) \underset{\sim}{W}^{*}_{bd}\left(\frac{t}{\beta} - \tfrac{1}{2}\tau(\alpha+\mu), \frac{f}{\beta} - \tfrac{1}{2}\nu(\alpha+\mu)\right), \quad \text{(A-8)}$$

where we used (61). Thus, this very general two-dimensional convolution and Fourier transform of cross CAFs can be expressed as a product of two different cross WDFs. For $\beta=\beta=\frac{1}{2}$, $\alpha=\alpha=1$, $\mu=\mu=1$, this result reduces to (81).

As a third case, let $\gamma=\beta$, $\gamma=-\beta$. There follows a two-dimensional relation involving both a convolution and a correlation:

$$\iint d\nu' d\tau' \exp(i2\pi\nu' t - i2\pi f\tau')\ X_{ab}(\beta\nu'+\alpha\nu, \beta\tau'+\alpha\tau)\ X_{cd}^{*}(-\beta\nu'+\mu\nu, \beta\tau'+\mu\tau)$$

$$= |\beta\beta|^{-1} \exp\left(+i2\pi f\tau\frac{\alpha+\mu}{2\beta} - i2\pi\nu t\frac{\alpha-\mu}{2\beta}\right) \times$$

$$\times\ \underset{\sim}{W}_{ad*}\left(\frac{t}{\beta} + \tfrac{1}{2}\tau(\alpha-\mu), \frac{f}{\beta} + \tfrac{1}{2}\nu(\alpha+\mu)\right)\ \underset{\sim}{W}_{bc*}^{*}\left(\frac{t}{\beta} - \tfrac{1}{2}\tau(\alpha-\mu), \frac{f}{\beta} - \tfrac{1}{2}\nu(\alpha+\mu)\right), \quad \text{(A-9)}$$

where $\underset{\sim}{W}(t,f) = \tfrac{1}{2}W(\tfrac{1}{2}t, \tfrac{1}{2}f)$ again. Observe the conjugates on subscripts d and c of the scaled WDFs $\underset{\sim}{W}$.

For $\beta=\beta=\tfrac{1}{2}$, $\alpha=\alpha=1$, $\mu=\mu=1$, this relation becomes

$$\iint d\nu' d\tau'\ \exp(i2\pi\nu' t - i2\pi f\tau')\ X_{ab}(\nu+\tfrac{1}{2}\nu', \tau+\tfrac{1}{2}\tau')\ X_{cd}^{*}(\nu-\tfrac{1}{2}\nu', \tau+\tfrac{1}{2}\tau') =$$

$$= 4\ \exp(i4\pi f\tau)\ \underset{\sim}{W}_{ad*}(2t, 2f+\nu)\ \underset{\sim}{W}_{bc*}^{*}(2t, 2f-\nu) =$$

$$= \exp(i4\pi f\tau)\ W_{ad*}(t, f+\tfrac{1}{2}\nu)\ W_{bc*}^{*}(t, f-\tfrac{1}{2}\nu)\ . \quad \text{(A-10)}$$

APPENDIX B - PRODUCTS OF WDFs

In this appendix, we will also generalize the results in (81) and (84), but now for products of two WDFs, to allow for more general arguments. Again, we begin by considering general two-dimensional functions as in figure 1. In particular, let

$$g(t) = W_1(t,f_a) \ , \quad h(t) = W_2(t,f_b) \ , \tag{B-1}$$

in (4). Then

$$G(\nu) = \Phi_1(\nu,f_a) \ , \quad H(\nu) = \Phi_2(\nu,f_b) \ , \tag{B-2}$$

giving

$$\int dt' \exp(-i2\pi\nu t') \, W_1(\beta t'+\alpha t,f_a) \, W_2^*(\gamma t'+\mu t,f_b) = \exp\left(i2\pi\nu t\frac{\alpha\gamma+\beta\mu}{2\beta\gamma}\right) \times$$

$$\times \int d\nu' \exp\Big(i2\pi\nu' t(\alpha\gamma-\beta\mu)\Big) \, \Phi_1\left(\gamma\nu'+\frac{\nu}{2\beta},f_a\right) \Phi_2^*\left(\beta\nu'-\frac{\nu}{2\gamma},f_b\right). \tag{B-3}$$

Now, let $f_a=\boldsymbol{\beta}f'+\boldsymbol{\alpha}f$, $f_b=\boldsymbol{\gamma}f'+\boldsymbol{\mu}f$, where the boldface constants are unrelated to their counterparts; that is, $\boldsymbol{\beta}$ need not equal β, with the same true of $\boldsymbol{\alpha},\boldsymbol{\mu},\boldsymbol{\gamma}$. Then Fourier transform (B-3) on f', to obtain

$$\iint dt'df' \exp(-i2\pi\nu t'+i2\pi f'\tau) \, W_1(\beta t'+\alpha t,\boldsymbol{\beta}f'+\boldsymbol{\alpha}f) \, W_2^*(\gamma t'+\mu t,\boldsymbol{\gamma}f'+\boldsymbol{\mu}f) =$$

$$= \exp\left(i2\pi\nu t\frac{\alpha\gamma+\beta\mu}{2\beta\gamma}\right) \iint d\nu' df' \exp\Big(+i2\pi\nu' t(\alpha\gamma-\beta\mu)+i2\pi f'\tau\Big) \times$$

$$\times \Phi_1\left(\gamma\nu'+\frac{\nu}{2\beta},\boldsymbol{\beta}f'+\boldsymbol{\alpha}f\right) \Phi_2^*\left(\beta\nu'-\frac{\nu}{2\gamma},\boldsymbol{\gamma}f'+\boldsymbol{\mu}f\right) . \tag{B-4}$$

In general, we cannot proceed any further on this double integral of a product of general two-dimensional functions W_1 and W_2.

Now let R_1 and R_2 be TCFs; that is,

$$R_1(t,\tau) = a(t+\tfrac{1}{2}\tau)\ b^*(t-\tfrac{1}{2}\tau) = R_{ab}(t,\tau) \ ,$$
$$R_2(t,\tau) = c(t+\tfrac{1}{2}\tau)\ d^*(t-\tfrac{1}{2}\tau) = R_{cd}(t,\tau) \ . \qquad \text{(B-5)}$$

Then Φ_1 and Φ_2 become SCFs:

$$\Phi_1(\nu,f) = \Phi_{ab}(\nu,f) = A(f+\tfrac{1}{2}\nu)\ B^*(f-\tfrac{1}{2}\nu) \ ,$$
$$\Phi_2(\nu,f) = \Phi_{cd}(\nu,f) = C(f+\tfrac{1}{2}\nu)\ D^*(f-\tfrac{1}{2}\nu) \ . \qquad \text{(B-6)}$$

Substitution in (B-4) yields

$$\iint dt'df' \exp(-i2\pi\nu t'+i2\pi f'\tau)\ W_{ab}(\beta t'+\alpha t,\beta f'+\alpha f)\ W_{cd}^*(\gamma t'+\mu t,\gamma f'+\mu f)$$

$$= \exp\left(i2\pi\nu t\frac{\alpha\gamma+\beta\mu}{2\beta\gamma}\right) \iint d\nu' df' \ \exp\left(+i2\pi\nu' t(\alpha\gamma-\beta\mu)+i2\pi f'\tau\right) \ \times$$

$$\times\ A\left(\beta f'+\alpha f+\tfrac{1}{2}\gamma\nu'+\tfrac{1}{4}\nu/\beta\right)\ B^*\left(\beta f'+\alpha f-\tfrac{1}{2}\gamma\nu'-\tfrac{1}{4}\nu/\beta\right) \ \times$$

$$\times\ C^*\left(\gamma f'+\mu f+\tfrac{1}{2}\beta\nu'-\tfrac{1}{4}\nu/\gamma\right)\ D\left(\gamma f'+\mu f-\tfrac{1}{2}\beta\nu'+\tfrac{1}{4}\nu/\gamma\right) \ . \qquad \text{(B-7)}$$

As a first case, let $\gamma=\beta$ and $\gamma=\beta$. Then (B-7) becomes

$$\iint dt'df' \exp(-i2\pi\nu t'+i2\pi f'\tau)\; W_{ab}(\beta t'+\alpha t, \beta f'+\alpha f)\; W^*_{cd}(\beta t'+\mu t, \beta f'+\mu f)$$

$$= \exp\left(i2\pi\nu t\frac{\alpha+\mu}{2\beta}\right) \iint d\nu' df' \; \exp\left(+i2\pi\nu' t\beta(\alpha-\mu)+i2\pi f'\tau\right) \times$$

$$\times\; A\left(\beta f'+\alpha f+\tfrac{1}{2}\beta\nu'+\tfrac{1}{4}\nu/\beta\right)\; B^*\left(\beta f'+\alpha f-\tfrac{1}{2}\beta\nu'-\tfrac{1}{4}\nu/\beta\right) \times$$

$$\times\; C^*\left(\beta f'+\mu f+\tfrac{1}{2}\beta\nu'-\tfrac{1}{4}\nu/\beta\right)\; D\left(\beta f'+\mu f-\tfrac{1}{2}\beta\nu'+\tfrac{1}{4}\nu/\beta\right) =$$

$$= |\beta\beta|^{-1} \exp\left(+i2\pi\nu t\frac{\alpha+\mu}{2\beta} - i2\pi f\tau\frac{\alpha+\mu}{2\beta}\right) \times$$

$$\times\; X_{ac}\left(f(\alpha-\mu)+\frac{\nu}{2\beta}, t(\alpha-\mu)+\frac{\tau}{2\beta}\right)\; X^*_{bd}\left(f(\alpha-\mu)-\frac{\nu}{2\beta}, t(\alpha-\mu)-\frac{\tau}{2\beta}\right) . \quad \text{(B-8)}$$

Thus, the very general two-dimensional correlation and Fourier transform of cross WDFs can be expressed as a product of two different cross CAFs. For $\beta=\beta=1$, $\alpha=\alpha=\frac{1}{2}$, $\mu=\mu=-\frac{1}{2}$, this result reduces to (84).

As a second case, let $\gamma=-\beta$ and $\gamma=-\beta$. Then (B-7) becomes

$$\iint dt'df' \exp(-i2\pi\nu t'+i2\pi f'\tau)\; W_{ab}(\alpha t+\beta t', \alpha f+\beta f')\; W^*_{cd}(\mu t-\beta t', \mu f-\beta f')$$

$$= \exp\left(i2\pi\nu t\frac{\alpha-\mu}{2\beta}\right) \iint d\nu' df' \; \exp\left(-i2\pi\nu' t\beta(\alpha+\mu)+i2\pi f'\tau\right) \times$$

$$\times\; A\left(\beta f'+\alpha f-\tfrac{1}{2}\beta\nu'+\tfrac{1}{4}\nu/\beta\right)\; B^*\left(\beta f'+\alpha f+\tfrac{1}{2}\beta\nu'-\tfrac{1}{4}\nu/\beta\right) \times$$

$$\times\; C^*\left(-\beta f'+\mu f+\tfrac{1}{2}\beta\nu'+\tfrac{1}{4}\nu/\beta\right)\; D\left(-\beta f'+\mu f-\tfrac{1}{2}\beta\nu'-\tfrac{1}{4}\nu/\beta\right) =$$

$$= |\beta\beta|^{-1} \exp\left(+i2\pi\nu t\frac{\alpha-\mu}{2\beta} - i2\pi f\tau\frac{\alpha-\mu}{2\beta}\right) \times$$

$$\times \underset{\sim}{W}_{ac}\left(t(\alpha+\mu)+ \frac{\tau}{2\beta}, f(\alpha+\mu)+ \frac{\nu}{2\beta}\right) \underset{\sim}{W}^{*}_{bd}\left(t(\alpha+\mu)- \frac{\tau}{2\beta}, f(\alpha+\mu)- \frac{\nu}{2\beta}\right), \quad (B\text{-}9)$$

using (61). Thus, the very general two-dimensional convolution and Fourier transform of cross WDFs can be expressed as a product of two different cross WDFs. For $\beta=\beta=½$, $\alpha=\alpha=1$, $\mu=\mu=1$, this result reduces to (81).

For $\tau = 0$, $\nu = 0$, $b(t) = a(t)$, $d(t) = c(t)$, (B-9) reduces to

$$\iint dt'df' \; W_{aa}(\alpha t+\beta t', \alpha f+\beta f') \; W_{cc}(\mu t-\beta t', \mu f-\beta f') =$$

$$= |\beta\beta|^{-1} \left|\underset{\sim}{W}_{ac}\left(t(\alpha+\mu), \; f(\alpha+\mu)\right)\right|^2, \quad (B\text{-}10)$$

which is nonnegative for all parameter values and waveforms a(t) and c(t). This is a generalization of (96).

APPENDIX C - A GENERALIZED WDF

When a signal is reflected from a moving target, the effect is to contract (or expand) the time scale of the echo, rather than cause a frequency shift. This requires us to consider a more general version of a WDF. To begin, if waveforms

$$\underset{\sim}{a}(t) \equiv a(\alpha t) \ , \quad \underset{\sim}{b}(t) \equiv b(\alpha t) \ , \quad \alpha > 0 \ , \tag{C-1}$$

then their cross WDF is

$$W_{\underset{\sim}{a}\underset{\sim}{b}}(t,f) = \frac{1}{\alpha} W_{ab}(\alpha t, f/\alpha) \ . \tag{C-2}$$

Thus, we have need to consider integrals of the form

$$K \equiv \iint dt' df' \exp(-i2\pi\nu t' + i2\pi f'\tau) \ W_{ab}(t',f') \ W_{cd}^{*}(t-\alpha t', f-f'/\alpha) . \tag{C-3}$$

This form is general enough to accommodate integrand

$$W_{ab}(\beta t', \beta f') \ W_{cd}^{*}(t-\alpha t', f-f'/\alpha) \tag{C-4}$$

by a change of variable.

To accomplish evaluation of (C-3), we must define a generalized WDF as

$$W_{ab}(t,f;p) \equiv \int d\tau \exp(-i2\pi f\tau) \ a(t+p\tau) \ b^{*}(t-(1-p)\tau) \ . \tag{C-5}$$

Then we have the usual WDF as a special case, namely

$$W_{ab}(t,f;\tfrac{1}{2}) = W_{ab}(t,f) \ . \tag{C-6}$$

Also, (C-5) enables us to evaluate the following more general integral according to

$$\int dt' \exp(-i2\pi ft')\, a(t')\, b^*(t-\alpha t') =$$

$$= p \exp(-i2\pi ftp)\, W_{ab}(pt,pf;p) \;; \quad p = \frac{1}{1+\alpha} \,. \qquad \text{(C-7)}$$

Now we are in a position to reconsider integral K defined above in (C-3):

$$K = \iint dt'df' \exp(-i2\pi\nu t'+i2\pi f'\tau) \int du \exp(-i2\pi f'u)\, a(t'+\tfrac{1}{2}u) \times$$

$$\times\, b^*(t'-\tfrac{1}{2}u) \int dv \exp[i2\pi(f-f'/\alpha)v]\, c^*(t-\alpha t'+\tfrac{1}{2}v)\, d(t-\alpha t'-\tfrac{1}{2}v). \quad \text{(C-8)}$$

The integral on f′ yields $\delta(\tau-u-v/\alpha)$. Integration on u then yields

$$K = \iint dt'dv \exp(-i2\pi\nu t'+i2\pi fv)\, a(t'+\tfrac{1}{2}\tau-\tfrac{1}{2}v/\alpha) \times$$

$$\times\, b^*(t'-\tfrac{1}{2}\tau+\tfrac{1}{2}v/\alpha)\, c^*(t-\alpha t'+\tfrac{1}{2}v)\, d(t-\alpha t'-\tfrac{1}{2}v) \,. \qquad \text{(C-9)}$$

Now let

$$x = t'+\tfrac{1}{2}\tau-\tfrac{1}{2}v/\alpha \,, \quad y = t'-\tfrac{1}{2}\tau+\tfrac{1}{2}v/\alpha \,;$$

$$t' = \tfrac{1}{2}(x+y) \,, \quad v = \alpha(y-x+\tau) \,. \qquad \text{(C-10)}$$

The Jacobian of this two-dimensional transformation is α, leading to

$$K = \alpha \iint dx\, dy\, \exp[-i\pi\nu(x+y)+i2\pi f\alpha(y-x+\tau)] \times$$

$$\times\, a(x)\, b^*(y)\, c^*(t+\tfrac{1}{2}\alpha\tau-\alpha x)\, d(t-\tfrac{1}{2}\alpha\tau-\alpha y) =$$

$$= \alpha \exp(i2\pi\alpha f\tau) \int dx\, \exp[-i2\pi(\alpha f+\tfrac{1}{2}\nu)x]\, a(x)\, c^*(t+\tfrac{1}{2}\alpha\tau-\alpha x) \times$$

$$\times \int dy\, \exp[i2\pi(\alpha f-\tfrac{1}{2}\nu)y]\, b^*(y)\, d(t-\tfrac{1}{2}\alpha\tau-\alpha y) =$$

$$= \frac{\alpha}{(1+\alpha)^2} \exp\left(i2\pi\frac{\alpha f\tau-\nu t}{1+\alpha}\right) W_{ac}\left(\frac{t+\frac{1}{2}\alpha\tau}{1+\alpha}, \frac{\alpha f+\frac{1}{2}\nu}{1+\alpha}; \frac{1}{1+\alpha}\right) \times$$

$$\times\, W_{bd}^*\left(\frac{t-\frac{1}{2}\alpha\tau}{1+\alpha}, \frac{\alpha f-\frac{1}{2}\nu}{1+\alpha}; \frac{1}{1+\alpha}\right) , \qquad \text{(C-11)}$$

by use of (C-7). For $\alpha = 1$, this reduces to alternative form (83), upon use of (C-6) and (61).

REFERENCES

[1] H. L. Van Trees, **Detection, Estimation, and Modulation Theory, Part III, Radar-Sonar Signal Processing and Gaussian Signals in Noise**, John Wiley & Sons, Inc., New York, NY, 1971.

[2] A. H. Nuttall, **Wigner Distribution Function: Relation to Short-Term Spectral Estimation, Smoothing, and Performance in Noise**, NUSC Technical Report 8225, Naval Underwater Systems Center, New London, CT, 16 February 1988.

[3] J. E. Moyal, "Quantum Mechanics as Statistical Theory", **Proceedings Cambridge Philosophical Society**, volume 45, 1949.

[4] A. H. Nuttall, **Alias-Free Wigner Distribution Function and Complex Ambiguity Function for Discrete-Time Samples**, NUSC Technical Report 8533, Naval Underwater Systems Center, New London, CT, 14 April 1989.

[5] A. H. Nuttall, **The Wigner Distribution Function With Minimum Spread**, NUSC Technical Report 8317, Naval Underwater Systems Center, New London, CT, 1 June 1988.